Roller Coaster Math

Slopes and Areas

Calculus Fundamentals

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"Elementary mathematics is simple but challenging. It is beautiful but baffling to many. It causes anxiety when it should cause excitement."
—Vera Sarina, The Living Tree of Mathematics, NCTM 2021

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Front cover: The constant velocity of the roller-coaster going up the first straight incline is found using arithmetic: velocity = distance divided by time. Coming down, the roller coaster is in free fall due to gravity and velocity is not constant. Finding the velocity at any instant in time now must be found another way.

Contents

Introduction

The Story of Galileo and the Beginning of Calculus

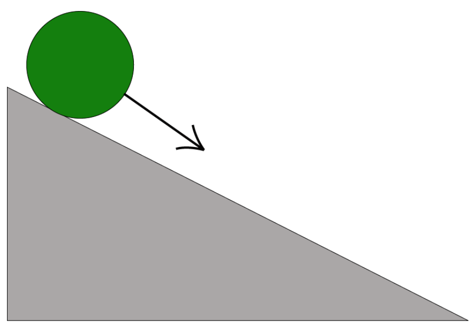
Galileo was a famous mathematician who lived more than 300 years ago. He formulated the laws that govern the motion of objects in free fall. He also made many other significant discoveries in physics and astronomy. Albert Einstein called him the “father of modern science”. NASA even named a spacecraft after him.

Galileo was a good student and an accomplished musician. At first, he wanted to become a doctor, but he never finished college. He was fascinated by mathematics and learned most of it on his own. Galileo was especially interested in finding mathematical concepts that described nature’s behavior.

One of his discoveries was to become the most useful in all of science. His experiments showed how gravity acts on a freely falling object. He started by dropping a ball from the top of a tower and noticed that the ball gained speed as it moved downward. The maximum speed would be at the instant the ball hit the ground.
(See Appendix 2 for a graph of the ball's path)

From this experiment, Galileo concluded that it is not possible to multiply the velocity by the time to find the distance traveled by the dropped ball. This would give the correct distance only if the velocity were constant. His experiment proved that **velocity is not constant for an object in free fall** (with air resistance neglected). How did he determine the numbers?

During his time as professor of mathematics at the University of Pisa, Galileo devised a method to better see the free fall motion of the ball. He would roll a ball down an inclined plane to better visualize what should happen in vertical fall. He wanted to determine the relationship between the time and distance traveled.



Galileo measured the distance a ball covered over larger and larger time periods. The purpose of the incline was to slow the speed of the ball. This would slow the rate at which velocity changes. He then could carefully observe and collect data about the motion. This would allow him to make accurate measurements of time intervals.

By gradually increasing the slope of the incline, he made an important conclusion about freely falling objects: the motion of a ball on an inclined plane is equivalent to a ball in vertical free fall. He discovered that objects under free fall motion have a continuously changing velocity. The ball went faster every moment while the rate of velocity change remained constant.

His experiment showed that the distance a body falls under the force of gravity, and gravity alone is proportional to the time of the fall squared. Today we know that relation to be $y = 16t^2$.

Galileo also discovered that anything falling to the surface of the earth has the same constant rate of velocity change. (This assumes that there is no air resistance). This is known as **Acceleration.** Any object gains velocity at the same rate of 32 feet per second each second. In symbols, if a = acceleration, then a = 32. So the ball dropped from the top of the cliff will start with zero velocity. At the end of one second, its velocity is 32 feet per second. At the end of two seconds, its velocity is 32 times 2, or 64 feet per second. This continues each second. At the end of t seconds, its velocity (v) is 32t feet per second—or in symbols, v=32t.

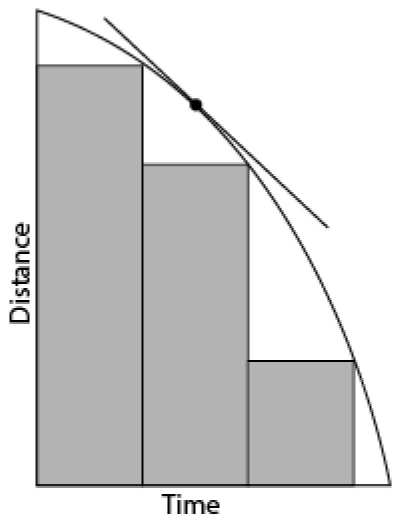
From this experiment, Galileo concluded that it is not possible to multiply the velocity by the time to find the distance traveled by the dropped ball. This would give the correct distance only if the velocity were constant. His experiment proved that **velocity is not constant for an object in free fall** (again, air resistance neglected).

Galileo developed the motion equations, but it was Isaac Newton and Gottfried Leibniz who discovered the fascinating relationship between them and ultimately developed what we now call **Calculus**.

We use **Calculus** to explain all physical phenomena.

It may be helpful before proceeding further to read
Appendix 1, page 38 : Need to Know

1 What is Calculus About



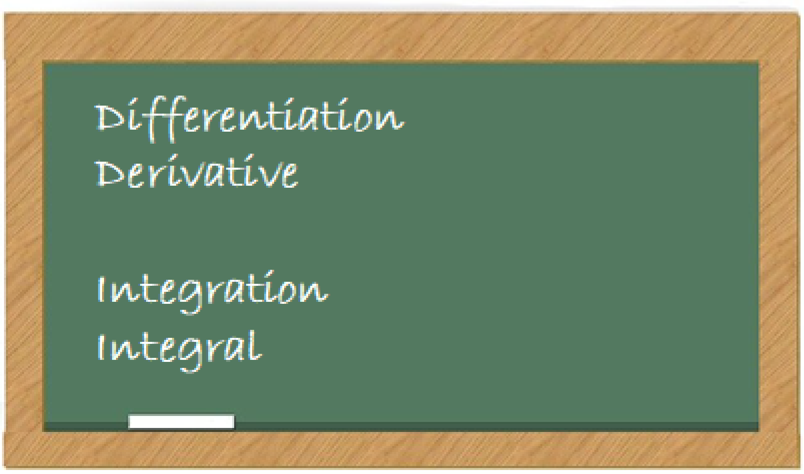
The roller coaster descends in free fall. Seconds later, a picture is taken. The challenge is to find the velocity of the roller coaster at that moment in time.

The graph of distance over time produces a curve which represents the path of the roller coaster. You can find the velocity at a point on the curve by finding the **steepness (known as Slope),** of a line that touches the curve at that tangent point. You can do this by looking at the roller coaster position through smaller and smaller intervals of time. Finding the **Slope** of the curve is called **Differentiation**. The **Slope** is the Calculus **Derivative.**

You can use rectangles to approximate the **area** under a curve. As the width of each inscribed rectangle is repeatedly reduced, the number of rectangles increases. When the width gets too small to measure—but never equals zero—you can find the actual **area** by adding the areas of all the rectangles. This process is called **Integration**. The result is called the **Integral.**

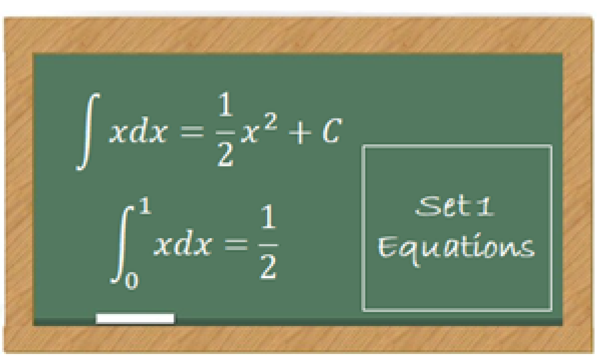
Roller Coaster Math presents the ideas of Calculus as a story. Here, two students and their teacher work together to understand the basic ideas of Calculus. They do this by analyzing the falling motion of a roller coaster. They discover that the Calculus **Derivative** is a way of measuring change at an instant. They learn that the area under a curve and between two points is the main idea for the Calculus **Integral**.

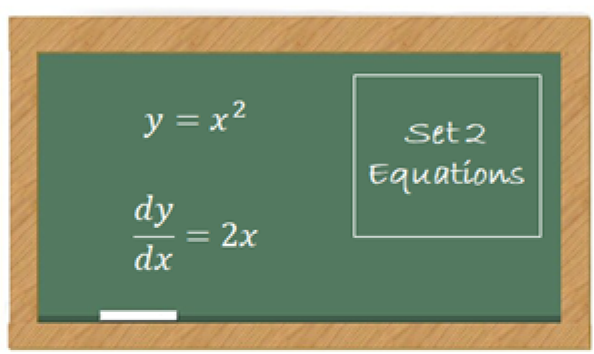
They also notice that just as addition and subtraction are **inverse** (opposite) operations, so too are **Differentiation** and **Integration**.



Two inverse operations

The goal is to understand two sets of equations.





2 How Fast Was the Roller Coaster Going?

A roller coaster slowly goes up to its highest point. Tensions are rising. Tyler and Taylor are excited—they know what is about to happen. Then, at the top, they hold on tight and prepare for the fast, exciting ride down.

The roller coaster begins its fall downward. It gains speed going faster and faster. Three seconds later, a camera mounted 144 feet from the beginning of the fall takes a picture, preserving the excited looks on their faces.

Later, they return to school. Their math teacher uses that exciting moment to help explain some important math concepts.

She shows them a figure of a roller coaster going up the first straight incline at a **constant** velocity. Then, she points to the roller coaster as it descends the first steep incline. She explains that it gains velocity going faster and faster. She asks the students to find the velocity of the roller coaster at the instant the picture was taken.

"No problem," Tyler explains, "velocity is distance divided by time. Velocity, therefore, is $\frac{144}{3}$= 48 $\frac{ft}{sec}$."

"No, no," Taylor exclaims, "that's not right! It doesn't make sense since the roller coaster goes faster as it goes down." The teacher thanks Taylor for her correct response.

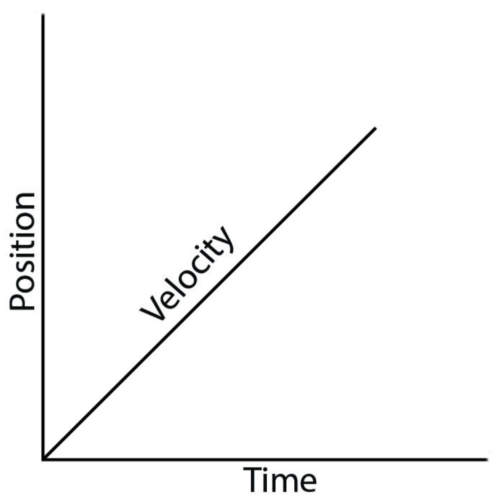
The teacher begins by explaining that the 48 $\frac{ft}{sec}$ velocity that Tyler found could only be valid only if velocity is constant as it is on the ride up the first steep incline. This is called **average velocity**, which equals a change in distance divide by a change in time: ($\frac{a\ change\ in\ distance}{a\ change\ in\ time}$).

Coming down the first incline, the roller coaster is in free fall—the velocity is not constant but ever increasing due to gravity.

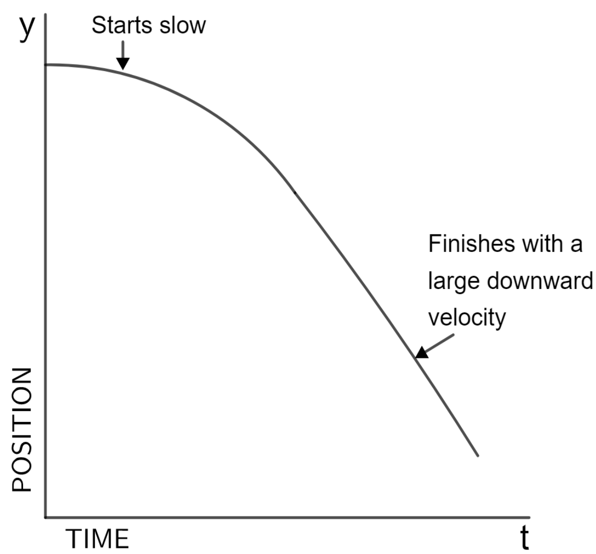
The teacher explains that a long time ago, a man named Galileo wanted to know how gravity affected the motion of a falling object. He discovered that the distance traveled by an object in free fall was proportional to time squared. That is, distance (y) is equal to some number multiplied by time (t) squared (t^2).
Note: "y" and "t" are variables. See Glossary: Variables

Galileo also discovered that an object in free fall gains velocity each second at a rate of 32 $\frac{ft}{sec}$. So at the end of one second, the velocity is 32 $\frac{ft}{sec.}$. At the end of two seconds, the velocity is (32·2) = 64$\frac{ft}{sec.}$. If v represents velocity, then, **v = 32t.** We know the time is 3 seconds, so the velocity at the picture point is

v(3) = 32(3) = **96 ft/sec** .



The Position-Time Graph illustrates a constant velocity. The slope of the line is equal to the velocity of the Roller Coaster as it goes up the first straight incline.



This Position-Time Graph Illustrates a changing velocity. Coming down, velocity is not constant because the roller coaster is in free fall. Velocity now is proportional to time squared (t^2), and represented by a curve. The slope of the curve is always changing (not constant) because the velocity of the roller coaster is always changing.

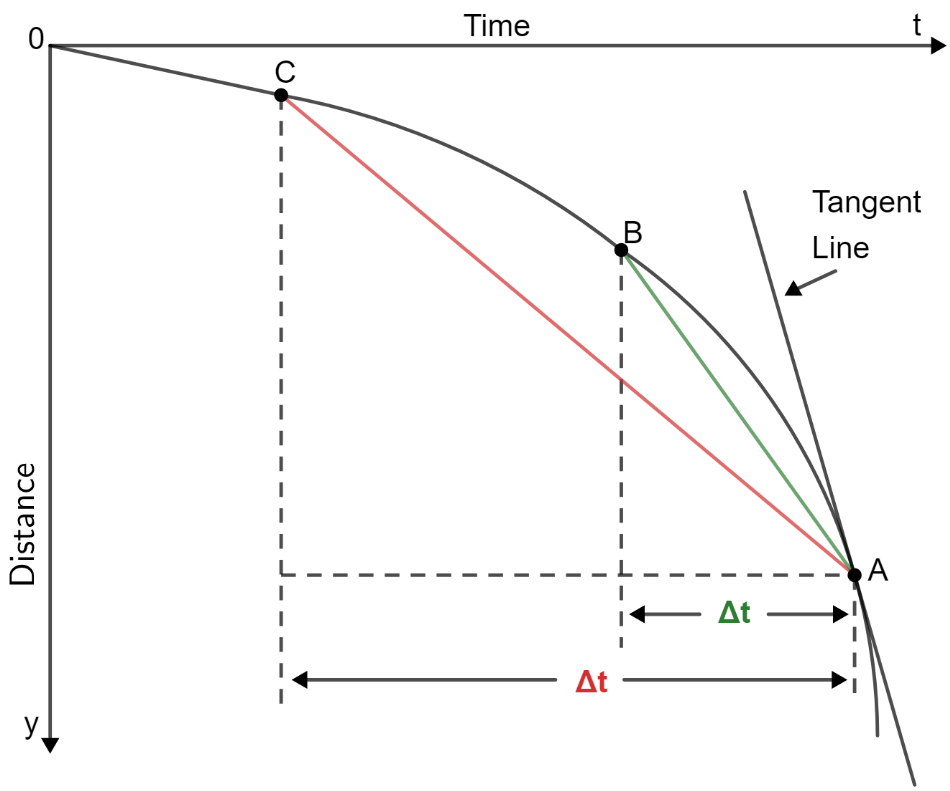
Key concept The early Greeks observed that the velocity of a falling object is always increasing (see Glossary: **Acceleration**). Galileo discovered that objects under free-fall motion have a changing velocity and a **constant** acceleration of 32 ft/sec² due to the force of gravity.

3 What's a Derivative? A Visual Approach

Velocity is the distance traveled divided by a period of time—and you can find it for **any** instant in time.

To find the velocity at a specific point, you look at the position of the roller coaster through smaller and smaller intervals of time. The teacher draws a picture to illustrate this.

Think of the curved line as the path of the falling roller coaster.



From the slope of a line to the slope of the tangent line that touches a curve at a point

The teacher asks the class to look at the red line from point C to point A. The slope of this line is $\frac{\Delta y}{\Delta t}$. It represents the average velocity between these two points.

Now, move down on the curve and look at the slope of the green line from point A to point B. The slope of this line is $\frac{\Delta y}{\Delta t}$. Notice that the green Δt (from A to B) is shorter than the red Δt (from A to C).

As points on the curve approach point A (where the camera was mounted), Δt gets very small. When Δt gets too small to measure, we replace it with **dt**. At the same time Δy is replaced by **dy**, $\frac{\Delta y}{\Delta t}$ becomes $\frac{dy}{dt}$. This represents the velocity at point A and we refer to it as the **derivative**.

The line that just touches the curve at point A—and at point A only—is called the **tangent line** and its slope is equal to the derivative (**velocity**) at that point. Note, however, that even though Δt gets very small, it never equals 0. You cannot divide by zero.

Therefore, as Δt gets smaller and smaller, the average velocity $\frac{\Delta y}{\Delta t}$ is replaced by the velocity at an instant in time symbol $\frac{dy}{dt}$.

So, $\frac{dy}{dt}$ **= 32t = instantaneous velocity = the derivative**
This is also known as **velocity at an instant in time**.

See Appendix 5, page 44, for a numerical, step-by-step approach for finding the derivative.

Analyzing average velocity $\frac{\Delta y}{\Delta t}$ over smaller and smaller values of Δ, finds the instant the picture was taken. The precise velocity will be found when the time between measurements is infinitely small.

Key concept: How to differentiate

Step 1: Divide a small change in one variable by a small change in another variable.
Step 2: Let these changes become very, very small until they approach zero.
Step 3: Find the value of the ratio between the two variables as they become infinitely small.

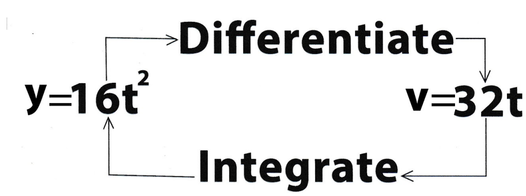
4 What's an Antiderivative? What's an Integral?

In the 17th century, two men named Isaac Newton and Gottfried Leibniz discovered that Galileo's distance and velocity equations had a fascinating relationship—an **inverse** relationship. This relationship defines two concepts of calculus: **differentiate** (find the derivative) and **integrate** (find the integral, also known as the antiderivative).

The teacher asks the students to look at the following figure. They see the velocity equation v = 32t on the right side and she reminds them that the distance equation is proportional to time squared.

They see a t^2 on the left side of the figure with a 16 in front of it. "Where did the 16 come from?" Kaleb asked. He would soon find out.

Find the derivative →



The connection between two equations: distance y = 16t^2 and velocity v = 32t

← Find the antiderivative

The teacher continues. To convert the velocity equation into the distance equation, you need to **integrate**. This is like undoing multiplication with division [8·2=16 and 16÷2=8].

How to Integrate — v = 32t = 32t^1

Step 1: Add one to the exponent: $32t^{1+1} = 32t^2$

Step 2: Divide by that number (2 in this case): $\frac{32}{2}t^2 \to 16t^2$

So $y = 16t^2$

“Oh,” says Kaleb. “There is the 16! That must be the number Galileo discovered to be equivalent to time squared.”

“Yes, Kaleb, you are correct,” replies the teacher.

The process used to find the distance equation is called **integration** and is also known as **antidifferentiation**. The symbol for integration is $\int$, an elongated S, and called the **integral** sign.

Distance (y) can now be defined as the **antiderivative** of velocity, which we write as: y = $\int$ 32t. We call the result of integration the **integral**. So, distance = the **integral** of velocity.

The teacher proves to the class that the distance to the picture point was indeed 144 ft if it took 3 seconds for the roller coaster to get there:

$$y = 16t^2 = 16(3^2) = 16(3)(3) = 144 \text{ ft}$$

If you want to go from the distance equation back to the velocity equation you need to **differentiate**.

How to differentiate y = 16t^2
(also see Appendix 4)

Step 1: Multiply 16 by the exponent → (2)16
Step 2: Take 1 away from the exponent → $32t^{2-1}$ = 32 t^1 = 32t = v

The process is called **differentiation**. It produces the **derivative**—the velocity at an instant in time—and is written as $\frac{dy}{dt}$, which means a very, very small change.

So, the derivative of y = 16t^2 is: $\frac{dy}{dt}$ = 32t = v

The class is now asked to differentiate v = 32t, which can be written as v = 32t^1.

Step 1: Multiple 32 by the exponent → (1)32
Step 2: Take 1 away from the exponent → 32t^{1-1} = 32t^0= 32

So, $\frac{dv}{dt}$= 32

This indicates that the derivative of velocity is a constant. This constant is called **acceleration**—the **rate** (how quickly) the speed or velocity changes over time. Slowing down and speeding up are examples of this.

Differentiation and Integration examples

y	$\frac{dy}{dt}$ **Derivative**	$\int ydt$ **Antiderivative**
1	0	t+C
t	1	$\frac{t^2}{2}$+ C
t^2	2t	$\frac{t^3}{3}$+C
t^3	$3t^2$	$\frac{t^4}{4}$+C
t^4	$4t^3$	$\frac{t^5}{5}$+C

Kaleb looks at this table and is puzzled. "Where did the +C come from?" he asks.

The teacher continues with an example to explain why we need to add a +C when we integrate.

The +C represents an unknown constant. Suppose we have the following equation: y = t^2+ 3. The derivative is: $\frac{dy}{dt}$= 2t + 0.

Kaleb now wants to know why there is a 0 in the equation. The teacher reminds the class that the derivative represents a change between variables. The constant 3 in the equation never changes, so the derivative will be 0.

If we **reverse** 2t by integration, we get $\frac{2}{2}t^{1+1}$ = t^2. We do *not* get back to the original equation. To get a complete integral, we need to add an unknown constant, which we show by writing +C. We need more information to find the exact value of C, so we leave it as a variable for now. So, the integral of 2t is t^2+ C.

Differentiation and Integration for the motion equations

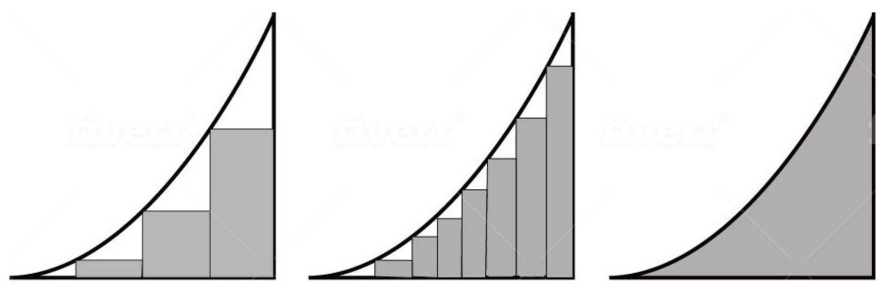
	Find the derivative y=16t^2	**Find the integral v=32t**
Step 1	Take the exponent and put it in front of t → $(2)16t^2$	Increase the power of t by 1 → $32t^{1+1}$
Step 2	Reduce the exponent by 1 and multiply numbers in front of t → $(2)16t^{2-1}$= (2)16t^1= 32t	To reverse the multiplication in differentiation, divide by the increased exponent → $32\frac{t^{(1+1)}}{1+1} = 32\frac{t^2}{2} = 16$
Step 3	So, 32t = $\frac{dy}{dt}$ = the **derivative** of position = velocity	So, $16t^2$=$\int 32t$ = the **integral** of velocity = position

Key concept Differentiation and integration are the reverse of each other in much the same way that subtraction is the reverse of addition, or division is the reverse of multiplication. See Glossary: Inverse operations.

5 Finding the Area Under a Curve

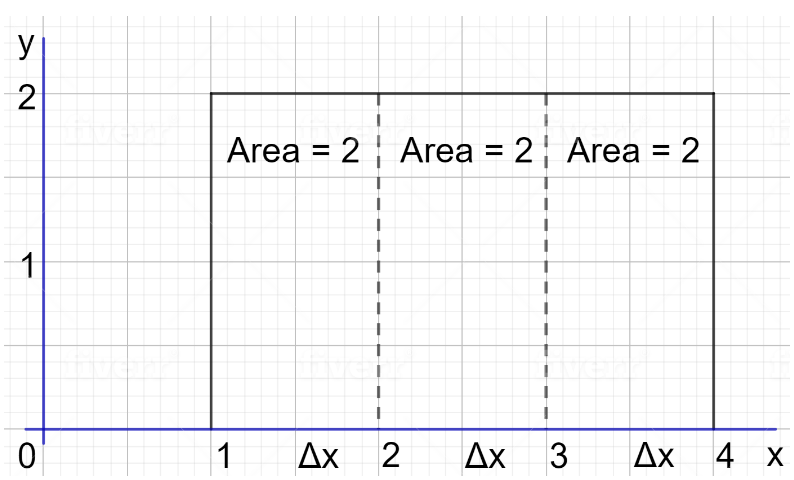
The teacher explains that it's easy to find the area of rectangles, circles, and triangles—but finding the area of some other shapes isn't so easy.

Consider, for example, that you need to find the area under a curve between set boundaries. You could find the area by dividing it into many small rectangles. By reducing the widths of each rectangle—that is, making each rectangle thinner—until they approach (but never reach) 0, and then adding them together, we get a better approximation of the area.



Approximating area under a curve using rectangles.

This process, also called **integration,** is a method for finding the area under a curve between set limits. The integration sign $\int$ is the symbol used for this process to express the "sum" of an infinite number of thin rectangles.



Three Rectangles
The teacher asks the class to look at the following figure. They notice that x is the variable instead of t.

The rectangle will be used to illustrate the **integration** concept.

A rectangle is divided into 3 parts. The total area of the rectangle is 6. The area of each part is 2. Δ**x = 1**. (Remember that Δ means a change. In this case, x is changing by 1.)

The area of the big rectangle is the base, 4 - 1 = 3, times the height, which is 2. This equals **6**. Each of the inside rectangles has an area of $2\Delta x$. Since $\Delta x = 1$, each inscribed rectangle has an area of 2(1) = 2.

So, the total area can be expressed as: $2(\Delta x) + 2(\Delta x) + 2(\Delta x) = 6$.

By allowing Δx to get smaller and smaller, the number of inscribed rectangles between x = 1 and x = 4 increases. When Δx gets too small to measure (but never equal to 0), it is replaced by "dx"—just like when we found the derivative.

Now the area can be expressed symbolically using the integration sign. This sign also means the "sum" of whatever follows it: So, $\int dx = x$ means that the sum of an infinite number of thin strips of width dx add up to x. When you integrate, the $\int$ sign and dx disappear and dx becomes x.

We can use all the information we have to write an equation to find the area under the curve. That equation looks like this:

$$A = \int_1^4 2dx$$

The numbers that are attached to the integration sign represent the distance between x = 1 and x = 4. They are called the **limits of integration** and are the right and left boundaries on the x axis.

As indicated in the Table on page 20, Row 1, the **antiderivative** (the result after doing integration) is 2x + C. Now, we **subtract the lower limit antiderivative from the upper limit antiderivative**: = (2·4+C) - (2·1 + C)

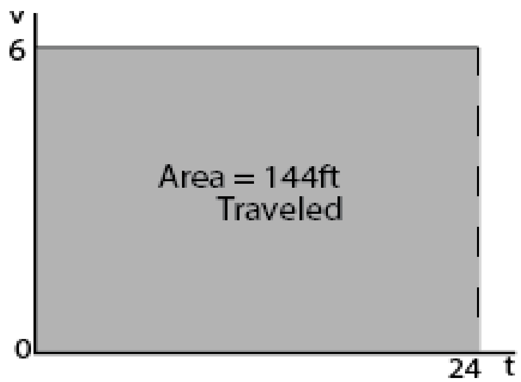
= (8+C - 2-C)

= (8 - 2 = 6)

The area under the rectangle is 6. Notice that the **two constants of integration cancel each other out**.

Example: Motion of the roller coaster demonstrates finding an area under a curve.

The teacher begins by explaining that the distance/time graph (see page 11) for the ride up the first steep slope can be expressed with velocity shown on the vertical axis in the following figure.



Assume the roller coaster is going up at a constant velocity of 6 $\frac{ft}{sec}$ for 24 seconds. Distance is velocity multiplied by time, so distance is (6)(24) = 144 ft. This is also the area of the rectangle. Area = (6)(24), and we can find this by integration:

A = $\int_{0}^{24} v\, dt = vt$ + C

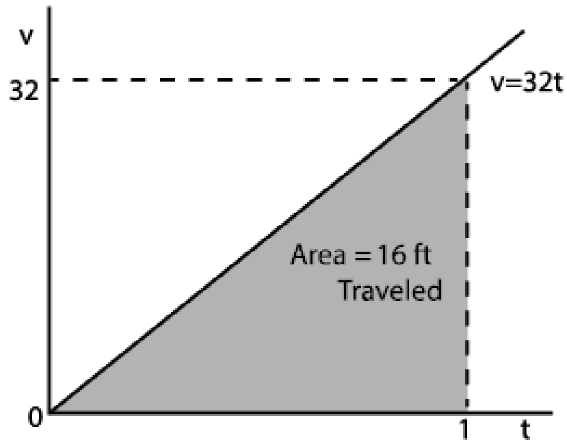
= (6·24 + C) - (0·24 + C)

= 144 + C - 0 - C

= 144

The area of the rectangle represents the distance the roller coaster traveled in 24 seconds. Notice, again, that the constants of integration cancel each other out.

Now the teacher asks the class to consider the following graph for v = 32t. This is called the **velocity triangle**. This is the graph of the velocity versus time for a falling object without considering air resistance. The area of the triangle represents the distance traveled.



We know the area of a triangle is one half the product of the base and the height:

$\frac{1}{2}(t)(v) = \frac{1}{2}(t)(32)$

With t = 1, area = **16**

We can also find this area by integration:

A = $\int_0^1 32t\,dt = \frac{32}{2}t^2 + C$

= (16·1 + C) - (0 + C)

= 16 + C - 0 - C = 16

We have discovered that the area under the graph between t = 0 and t = 1 represents the distance the roller coaster traveled in one second. Note, that 32 represents acceleration, which is the **slope** of the velocity triangle.

Key Concept We find the area under a curve between boundries x = A and x = B by using an integral evaluated over an interval. The process is to divide the area into a number of thin strips. By reducing the widths of the strips until they approach 0, we can get a better approximation of the area.

Two antiderivatives—one subtracted from the other—finds the area.

6 Application of the Derivative

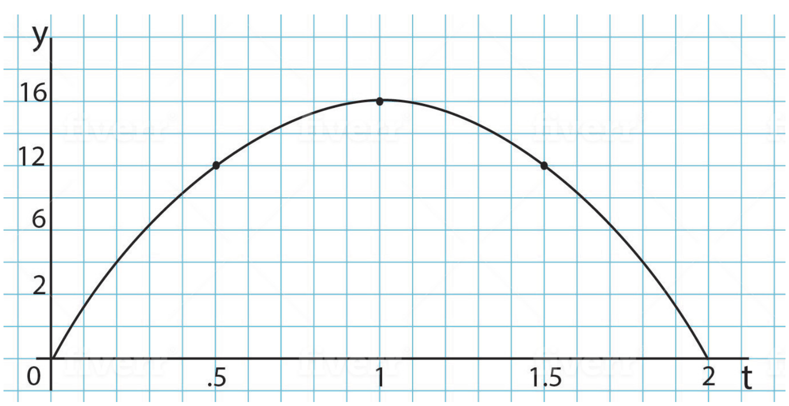
Key Concept When the derivative of a relationship between variables is equal to 0 (the tangent line at that point is horizontal), we can find the absolute maximum or minimum value of that relationship. This is an important application of differentiation.

Taylor wants to know why learning calculus is important and how it can help make hard problems simpler. The teacher asks the class to consider the following examples:

Example 1

The following figure represents the position of the roller coaster after the first free fall. y = distance and t = time. The roller coaster goes up and, at t = 1, comes back down.

Assume the equation for this curve is $y = 32t - 16t^2$. The graph shows that the maximum height of the roller coaster during this time period occurs at t = 1. The teacher asks the class to use their differentiation skills to confirm this.



Position of the roller coaster from time t = 0 to t = 2

Distance y is in feet, time t is in seconds.

The class knows that velocity is the derivative of distance. We can find the derivative of $y\ = 32t\ -\ 16t^{2}$, which we can write as y = $32t^{1}\ -\ 16t^{2}$.

Step 1. Multiply 32 by the exponent: 32(1) = 32
Step 2. Take 1 away from the exponent: $32t^{1-1}\ =\ 32$
Step 3. Multiply -16 by the exponent: (–16)(2)= –32
Step 4. Take 1 away from the exponent: $-32t^{2-1}\ =\ -32t^{1}\ =\ -32t$
Step 5. Combine steps 2 and 4 to get the derivative: $\frac{dy}{dt}$= 32 – 32t

At the instant the roller coaster goes from the upward motion to downward motion, the velocity is **zero**. So, for what value of t is the velocity equal to zero?

Setting the derivative to 0 will find the time to get to the highest point: 0 = 32 – 32t. Now find the value for t:

Step 1. Add -32 to both sides of the equation: –32 = –32 + 32 – 32t
Step 2. Simplify right side of equation: –32 = –32t
Step 3. Divide both sides of the equation by –32 to find t: $\frac{-32}{-32}\ =\ \frac{-32}{-32}$t
Therefore, t = 1

We find that it takes one second to get to the highest point. You find that point, by substituting 1 into the distance equation:
y $=\ 32(1)\ \ -\ 16(1^{2})$
y = 32 – 16
y = 16 ft

Example 2

The sum of two numbers is 6. How can we choose these numbers so that, when multiplied together, they produce the largest product possible. The teacher asked the class to try using numbers between 1 and 5 and to fill in the following table:

First number	**Second number**	**Product**

Finding the right combination using calculus

The sum of two numbers is 60. Let x = one number and 60–x = the other number. Let P = their product.

One way to find the right combination is by using trial and error. Create a table to find the right combination of numbers.

First Number	**Second number**	**Product**
10	50	500
20	40	800
30	30	900
40	20	800
50	10	500
x	**60 – x**	**Product** **P = x(60–x)**

From the table, it looks like the product is a maximum when each number is 30. The table does not account for any number between 10 and 20, like 7, or 3.2, or any other number.

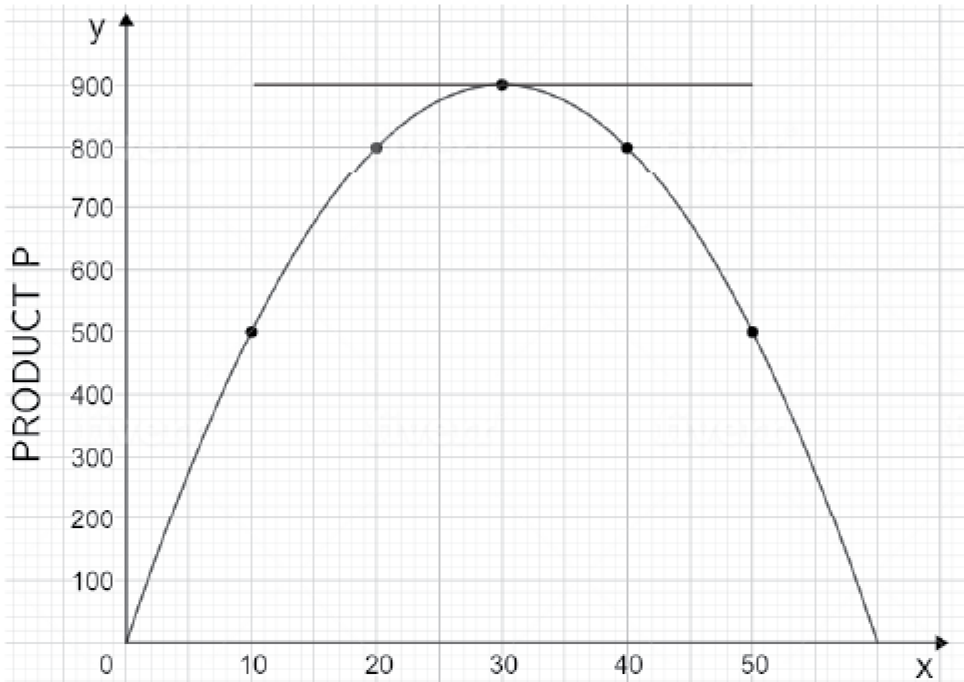
There is an easier way to do this. We can express the product P = x(60-x) as 60x-x^2 by multiplying x by (60–x).
Calculate the derivative of 60x-x^2, which we can express as $60x^1 - x^2$:

Step 1. Multiply the first number 60 by the exponent: (60)(1)= 60
Step 2. Take 1 away from the exponent $60x^{1-1} = 60x^0 = 60$
Step 3. Multiply the second number, 1, by the exponent: (1)(2)=2
Step 4. Take 1 away from the exponent $2x^{2-1} = 2x^1 = 2x$
Step 5. Combine results of steps 2 and 4 to find the derivative
$\frac{dP}{dx} = 60 - 2x$

We can find the maximum product when the derivative is equal to 0.
So, 60–2x = 0. Then, solve for x.

Step 1. Move –2x to other side of equation: 60 = 2x
Step 2. Divide both sides of the equation by 2: $\frac{60}{2} = \frac{2x}{2} = 30 = x$

Each number has a value of 30, which we can confirm by looking at the table and finding the number using calculus.



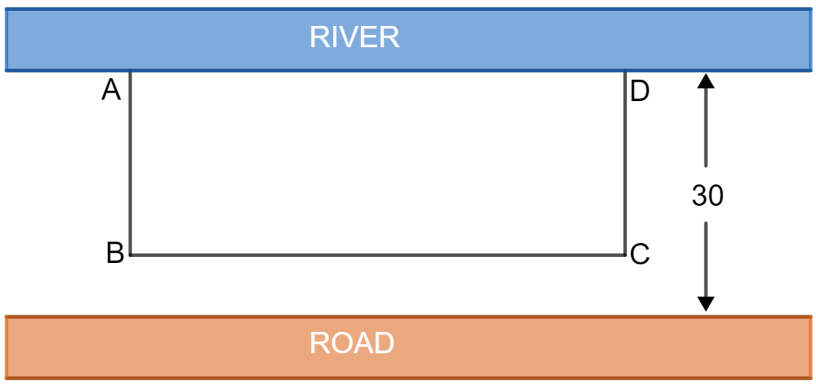
Graph of x and the product P

Maximizing the product of two numbers

We find the maximum when the slope of the tangent line touching the curve equals zero ($\frac{dP}{dx} = 0$).

See Glossary: Optimization and Maximum

Example 3
A farmer has 100 yards of fencing material. He wants to build an enclosure with an area as large as possible using all 100 yards of fencing.



The farmer's fence from A to D

You can see the boundaries above. At one end is a river and 30 yards away is a road. There is no need for a fence along the river edge.

At first, the farmer planned for a square enclosure, which would give him the maximum area. For a square, each of the three sides would be of equal length. But there is a problem. Dividing the 100 yards of fencing into 3 equal lengths gives $\frac{100}{3}$= 33 yards for each side. This would put part of the fence on the road!

So, a square enclosure is not possible. Therefore, the enclosed area must be a rectangular shape, as shown.

The farmer now has to decide how long the side AB should be. If, for instance, he decides to have side AB = 10 yards, then side CD also must be 10 yards long. That leaves 80 yards left for side BC. The fence will enclose an area of (80)(10)= 800 square yards. But is that the maximum it could be?

The farmer could make a table and come up with the desired dimension for side AB using trial and error. The following table shows that the maximum area is 1250 when AB and CD both = 25 and BC = 50.

Sides AB and CD	Side BC	Area y
10	80	800
20	60	1200
25	50	1250
30	40	1200
50	0	0
x	100-2x	Area y = 100x-2x^2

There is a faster way for finding the best value for AB that will also account for all possible side lengths. Let AB equal x, so CD is also equal to x. These two sides add up to 2x and therefore leave 100-2x for the length of BC. Let y equal the area in square yards. Now, area y = width times length = x(100-2x) = 100x - 2x^2.

The farmer wants to make y, the area, as large as possible. This will happen when the derivative of y is equal to 0, which happens when the tangent line has a slope of zero (*see Glossary, Optimization*).

So, for y = 100x-2x^2

$\frac{dy}{dx} = 100 - 4x$ = 0

Now solve the equation:

Add -100 to both sides of the equation: $-100 + 100 - 4x = -100$

Simplify the left side of the equation: $-4x = -100$

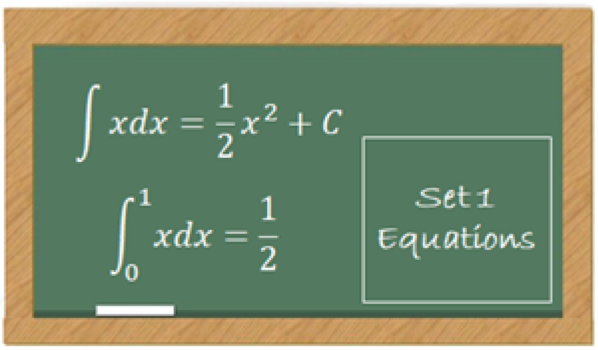
Divide both sides of the equation by -4: **x = 25**

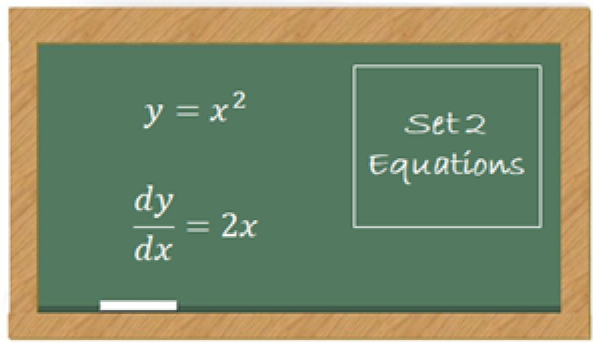
Therefore AB and CD are each 25 and
BC = 100 - 2x
= 100 - 2(25)
= 100 -50 = 50

The farmer has found the maximum possible area. It is (25)(50) = 1250 square yards.

7 Understanding Basic Concepts

The teacher asks the class to review the set 1 and 2 equations. (page 9)





One student excitedly raises her hand and says, "I know the meaning of the set 1 equations.These two equations represent the two concepts of **integration**!"

"Yes, you are right," the teacher proudly responds.

Two forms of integration

$$\int x dx = \frac{1}{2}x^2 + C$$

Antidifferentiation

$$\int_0^1 x dx = \frac{x^2}{2} \to \frac{1^2 - 0^2}{2} = \frac{1}{2}$$

Area under a curve

Another student exclaims, "Set 2 equations demonstrate **differentiation**!"

$y = x^2$ **Differentiate**: $\frac{dy}{dx} = 2x$

“Yes, you too are correct!” The teacher is very proud of them both.

Appendix 1 Need to Know

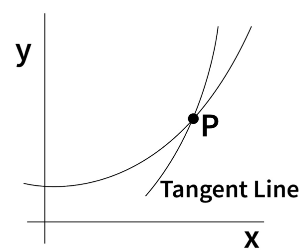
- The equal sign (=) means "the same as," not "the answer is"
- Distance = (rate) multiplied by (time)
- Average velocity is the distance traveled divided by the time of travel.

This can be expressed as: $\frac{\Delta\, distance}{\Delta\, time} = \frac{a\ change\ in\ distance}{a\ change\ in\ time}$. Here, the Greek letter Δ(delta) indicates **a small change or interval**. Note, however, that Δ in front of a quantity does not mean multiplying by that quantity.

- The slope of a line is the measure of the steepness of the line. As we go from one point on the line to another, the slope equals the vertical distance divided by the horizontal distance. A straight line has a constant slope.

See Appendix 4

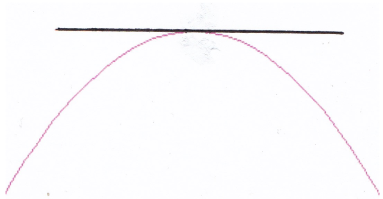
- An equation where the highest exponent of a variable is a 2 is called a **quadratic** equation—e.g., $y = 16t^2$. Here, y and t are variables and 16 is a constant. When graphed, a quadratic equation produces a curve that does *not* have a constant slope.
- A **tangent line** is a line that touches a curve at just one point. In the figure, the tangent line touches the curve at point **P**. The slope of a curve at a point is exactly the slope of the tangent line at that point.



- A variable is a symbol that can change value and is usually represented by a letter such as x, y , s, or t.
- The square root (symbol √) of a number is defined as the value, which gives the number when it is multiplied by itself.

You cannot divide by zero.

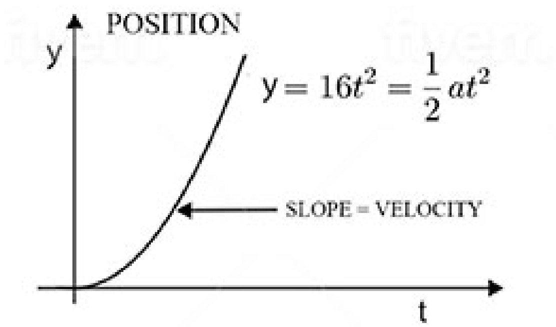
Key concept

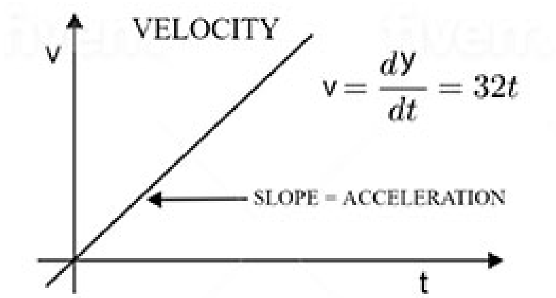


A horizontal tangent line has a slope of zero. This figure of a horizontal tangent line touching a curve at its highest point represents one of the most important applications of calculus.

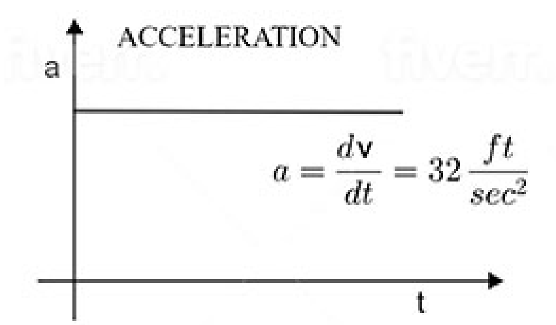
Appendix 2. Free Fall Motion in Six Graphs

Differentiation from position to acceleration





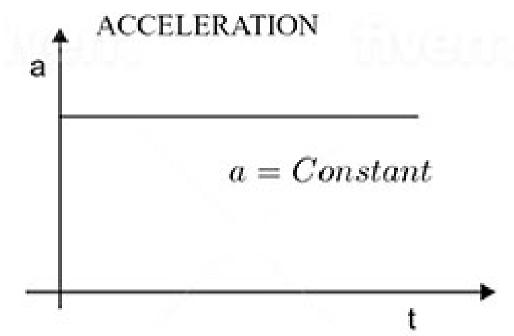
The derivative of the position is the slope of the position graph and also equals the velocity.

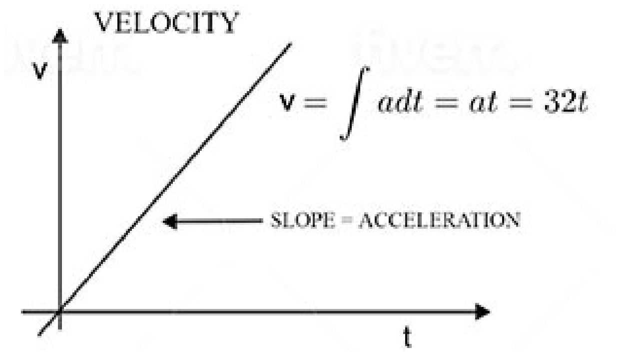


The derivative of the velocity is the slope of the velocity graph and also equals acceleration.

Free Fall in Reverse

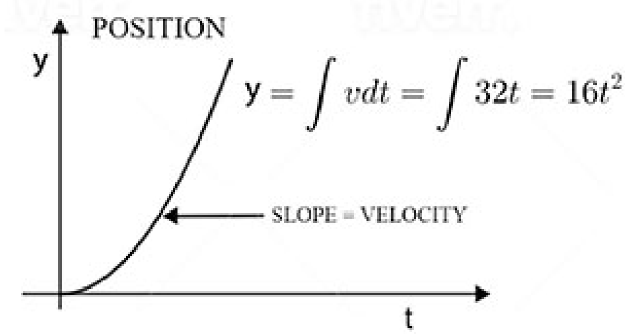
Integration from acceleration to position





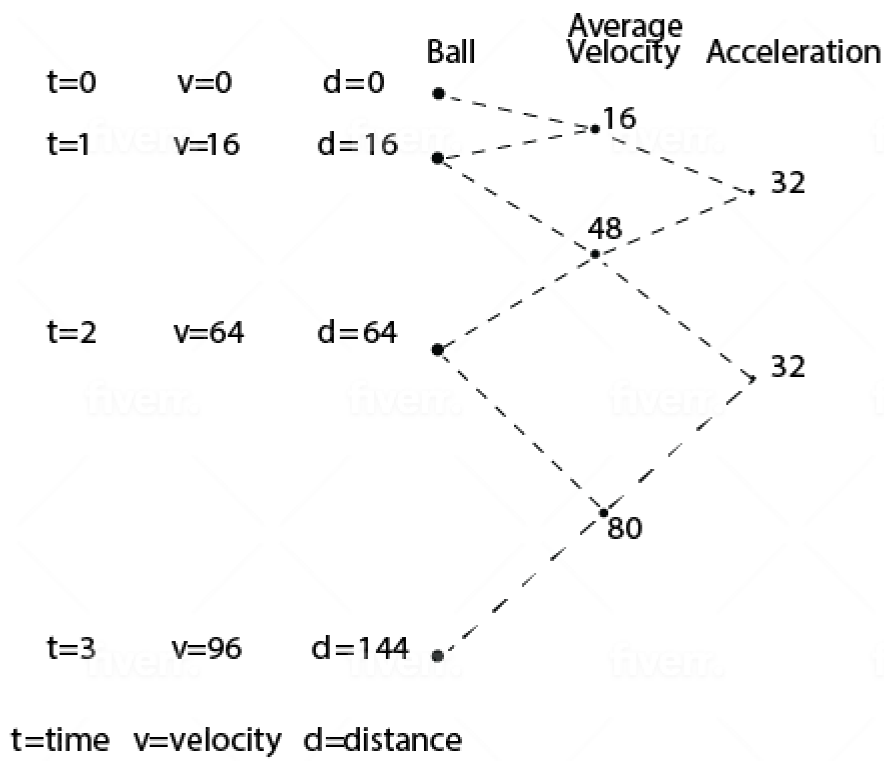
Integrating acceleration equals velocity.

Integrating velocity equals position.



Appendix 3. Free Fall Path to the Ground

Galileo dropped a ball from the top of a cliff. The following figure shows the path of the ball as it falls to the ground.

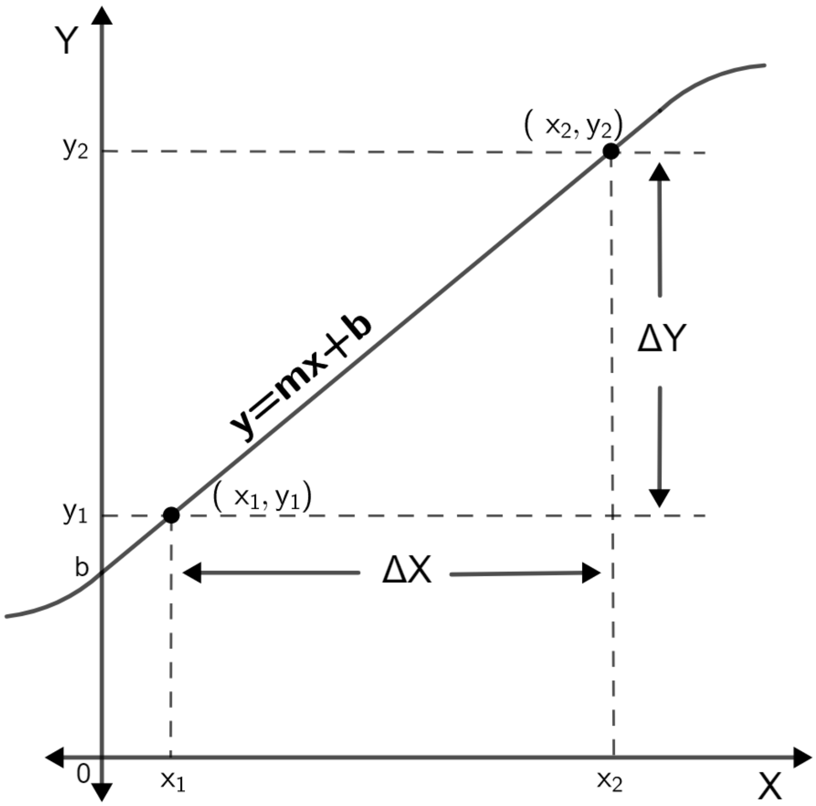


From the above figure, we can see the position of the ball at each second interval from t=0 to t=3. It also shows the average velocity at each interval. For example:

- After 1 second, the ball has fallen an average of 16 feet
- Between second 1 and second 2, the ball falls an average of 48 feet

Notice that as time increases, the difference in velocity between one-second intervals increases. Just to the right of the average velocity column is the column labeled “acceleration,” which represents the change in velocity over each three-second interval. Between time t=0 and t=1, the change in velocity is 32. Now notice that this change is the same during the next one-second interval. And, indeed, the change in velocity for each interval is 32. This is called the **constant of acceleration** due to gravity.

Appendix 4. The Slope of a Line and Average Velocity



Slope equals velocity

slope = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$

When the value of x increases, the value of y increases and the graph slants upward. The slope is therefore **positive.**

Note: When the value of x increases and the value of y decreases, the graph slants downward, resulting in a **negative** slope. When the value of x increases and the value of y remains constant, the graph is a horizontal line, resulting in **zero** slope.

Consider the x-y coordinate system in the figure above. The angled line represents the roller coaster incline. The steepness of the line is known as its **slope**. In the equation for a straight line, the letter "m" represents slope: y=mx+b, where b is the y coordinate (y intercept) when x = 0.

The slope m is equal to: $\frac{change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1}$ = the average rate of change.

A **change in distance** (expressed as Δy) divided by a **change in time** (expressed as Δt) represents the velocity. This only applies, however, when velocity is constant. On a graph, the distance plotted against time is a straight line with velocity represented by the slope (or steepness) of the line. Here, velocity is constant.

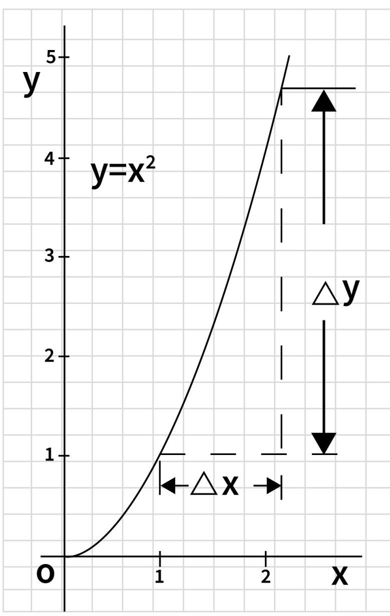
Finding the slope of a line

Step 1. Pick two points on the line and determine their coordinates.
Step 2. Determine the difference in the y-coordinates. This is known as the “rise.”
Step 3. Determine the difference in the x-coordinates. This is known as the “run.”
Step 4. Divide the difference in the y-coordinate by the difference in the x-coordinate: ($\frac{rise}{run}$). The result is called the “slope” of the line. On an x-y graph, $\frac{y_2 - y_1}{x_2 - x_1}$.

Appendix 5 Finding the Derivative Using Equations

Find the derivative for $y = x^2$ using the following method:

Step 1. Add a little to x. y will change too: $y + \Delta y = (x + \Delta x)^2$
Step 2. Apply binomial theorem: $y + \Delta y = x^2 + 2x\Delta x + \Delta x^2$
Step 3. Move y to the other side of the equation: $\Delta y = x^2 + 2x\Delta x + \Delta x^2 - y$
Step 4. Substitute x^2 for y: $\Delta y = x^2 + 2x\Delta x + \Delta x^2 - x^2$
Step 5. Simplify to eliminate x^2: $\Delta y = 2x\Delta x + \Delta x^2$
Step 6. Divide both sides by Δx: $\frac{\Delta y}{\Delta x} = \frac{2x\Delta x + \Delta x^2}{\Delta x}$
Step 7. Simplify right side of equation: $\frac{\Delta y}{\Delta x} = \frac{\Delta x(2x+\Delta x)}{\Delta x} = 2x + \Delta x$
As Δx approaches 0, the slope becomes a fixed value equal to 2x. Therefore, as Δx gets made smaller and smaller, $\frac{dy}{dx}$ replaces $\frac{\Delta y}{\Delta x}$, which represents the derivative: $\frac{dy}{dx}$= 2x



We can see the pattern again. It appears that the procedure is to multiply the quantity in front of t by the exponent and reduce the exponent by 1.

Graph of y =x^2

We can now state the rule for differentiation. This is known as the **power rule**. It is the basic technique for finding the derivative.

$y = t^n$ $\frac{dy}{dt} = nt^{n-1}$ For $y = x^n$ $\frac{dy}{dx}$= nx^{n-1}

Derivative examples

The derivative for y = t^n can be found by multiplying the quantity in front of t by the power of t (exponent) and then reducing the exponent by 1.

Note: $t = t^1$ and $1 = t^0$

Example 1

The derivative for t^5 is $\rightarrow \frac{dy}{dt} = 5t^4$

Example 2

The derivative for $t^2 + 5$ is $\rightarrow \frac{dy}{dt}$ = 2t

Note: The derivative of a number is 0.

Example 3

Differentiate y = 16t^2twice to find acceleration: $\frac{dy}{dt} = 32t$= velocity

$\frac{d}{dt}(\frac{dy}{dt}) = \frac{d^2y}{dt^2}$ = 32 = acceleration = **a**

Example 4

Find the instantaneous velocity (derivative) of a falling rock at the end of 2 seconds.

v =$\frac{dy}{dt}$= 32t = (32)(4) = 128 ft/sec

Example 5

Using the power rule, find the derivative of y with respect to x if

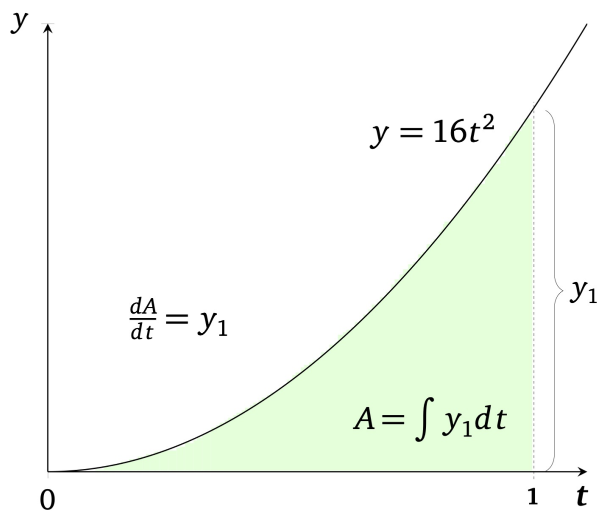
y = $x^3 \rightarrow \frac{dy}{dx} = 3x^2$

Example 6

Find the derivative of y with respect to x if y = 2$x^4$$\rightarrow \frac{dy}{dx} = 8x^3$

Appendix 6 The Fundamental Theorem of Calculus

Connecting the integral as sums with integrals as antiderivatives



Integrating to find the area A, and then differentiating the area to get back to y_1, which is the value of y at t=1.

Consider the equation **y =** $16t^2$ and the area A from t = 0 to t = 1, as graphed in the figure above. The following procedure will find the relationship between differentiation and integration. We don't have to apply boundaries, however, to understand the concept.

Step 1. Integrate to find the area A from t = 0 to t = 1:

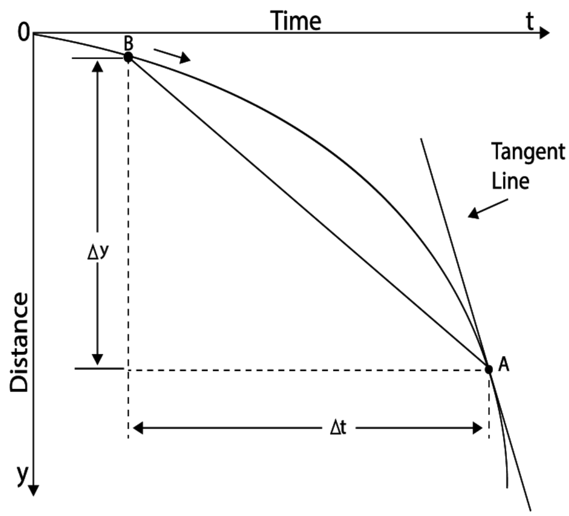
$$A = \int_0^1 16t^2\, dt = \left[two\ \ antiderivatives\ of\ 16(\frac{t^3}{3})\ \ evaluated\ over\ the\ interval\ 0, 1 \right]$$

Step 2. Find the derivative of the area: $\frac{dA}{dt} = \frac{d(16(\frac{t^3}{3}))}{dt} = 16(\frac{3t^2}{3}) = 16t^2 = y_1$

If the variables of the graph are t and y, then the derivative of the area under the graph at point t is equal to the y component at that point. The **derivative** of area gets us back to y. In other words, reversing differentiation is a way to find the area.

Appendix 7 Another way to find the derivative

You can find the velocity (derivative) of the roller coaster at the picture point. When you do, you have found the derivative.



The line from point B to point A in the figure represents the average velocity between these two points. The slope of this line is $\frac{\Delta y}{\Delta t}$. As point B on the curve approaches point A, Δt and Δy get smaller and smaller.

When Δt gets too small to measure, you replace it with dt ("d" means "an extremely small part of"). At the same time, you **replace** Δy **with dy**. Now $\frac{\Delta y}{\Delta t}$ becomes $\frac{dy}{dt}$. This represents the velocity at point A and we refer to it as the **derivative**.

The tangent line that just touches the curve at point A and at point A only has a slope that is equal to the velocity (**derivative**) at that point. **The line AB becomes the tangent line**.

Note, however, that even though Δt gets very small (to almost zero), it never equals exactly 0. You cannot divide by zero.

Recall that velocity v = 32t. So, v = $\frac{dy}{dt}$ = **32t = the derivative**.

$$\frac{dy}{dt} = \frac{a\ very\ small\ change\ in\ y}{a\ very\ small\ change\ in\ t}$$

The derivative will tell you **how fast you're going at any instant** in time.

Quiz

1. Differentiation and integration are inverse operations.
   True
   False

2. If y = distance and t = time, the symbol for the derivative is:
   A. dy·dt
   B. $\frac{dy}{dt}$
   C. $\frac{dt}{dy}$

3. The symbol for the Integral sign is:

   A. $\sum$

   B. I

   C. $\int$

4. Velocity at an instant in time is known as the:
   A. integral
   B. antiderivative
   C. derivative

5. Acceleration is the rate at which an object's velocity changes with time.
   True
   False

6. The integral of $2x^2$ is:
   A. $\frac{2}{3}x^3$
   B. $4x^2$
   C. x^3

7. The derivative of $2x^2$ is:
   A. 2x
   B. 4x
   C. $2x^3$

8. Distance is the antiderivative of velocity.
   True
   False

9. To find the maximum value in the equation of a curve, you have to set the derivative to:
   A. 1
   B. -1
   C. 0

10. The symbol for the sum of many little bits of t adding up to t is:

    A. $\int dt = t$

    B. $\int t = dt$

    C. $\int t\,dt = 0$

11. What does $\frac{dy}{dt}$ mean at any point on a curve?
    A. It represents the average velocity at that point
    B. It represents the maximum point of the curve
    C. It is the slope of the tangent line at that point

12. Show that $\int_{t=1}^{t=3} 5dt = 10$

13. An object is dropped from the top of a tall building.
    A. Find the exact velocity at 3 seconds.
    B. What is this velocity called?

14. Find $\frac{ds}{dt}$ when $s = 5t + 16t^2$.

15. Find $\frac{dy}{dx}$ when $y = x^8$.

16. Imagine dropping a rock from the top of a cliff that is 144 ft high. Find the time it takes to hit the ground.

17. Solve $\int_1^2 x^2 dx$.

18. Solve $\int_0^1 x dx$ and draw a graph to show that the integral is equal to $\frac{1}{2}$. The area of the graph is that of a:
A. Square
B. Triangle
C. Rectangle

19. How fast was the roller coaster going at the picture point if it takes 4 seconds to get there?

20. What is the antiderivative of X ? *(Don't forget the +C)*

21. Solve the equation $\frac{dy}{dx}$= 2x. That is, find y in terms of x. Show the steps to your solution. *(Hint: Multiply both sides of the equation by dx and then integrate.)*

Answer: y = $x^2 + C$

22. Show that the slope of the velocity triangle on page 25 is 32.

23. There is something missing in Appendix 2, From Acceleration to Position. What is it?

Glossary

Acceleration: Acceleration is the **rate** at which an object's velocity changes with time. Objects fall to earth with a constant acceleration of 32 ft/sec^2. (see Rate)

Algebra: In Arithmetic, we are dealing with numbers. In Algebra, we replace numbers with letters that represent a variable quantity and solve equations for a "particular" value of that quantity. (In calculus, we are interested in how a change in one variable affects another variable.)

Antiderivative: The inverse of the derivative. For example, the derivative of distance is velocity and the antiderivative of velocity is distance.

Area of a triangle: The area of a triangle is equal to one half the width of the base times the height.

Area under a curve—the limit of a sum: We find the area under the curve y=f(x) between x=A and x=B by using the definite integral: $\int_A^B f(x)\,dx$. The process is to divide the area into a number of thin strips. By reducing the widths of the strips until they approach 0, we can get a better approximation of the area.

Average: The sum of several quantities divided by the number of quantities.

Average rate of change: This indicates how rapidly a quantity changes (on average) over time. The change in the value of a quantity divided by the elapsed time or distance. For a function, this is the change in the y-value divided by the change in the x or t value. Example: Δ denotes a change or interval. Average rate of change = $\frac{\Delta y}{\Delta t}$.

Binomial theorem: An easy way to expand a binomial expression that has been raised to some power. Example: $(a+b)^2 = (a^2 + 2ab + b^2)$

Calculus: Calculus is the study and measurement of quantities that are changing. Basic concepts are application of derivatives and integrals (antiderivatives). In *Roller Coaster Math*, we find the derivative (velocity at an instant in time) by measuring shorter and shorter time intervals. Calculus is a way of measuring motion.

Calculus symbols:

$\int$ — The tall S is a symbol for an integral, which is the "sum" composed of an "infinite" number of infinitely thin rectangles exactly measuring the area under a curve.

Δ — Delta is a Greek letter used to denote a change or interval and is used to introduce the concept of the derivative. For example, $\Delta y = y_2 - y_1$ means a change in y.

dx — This is called the differential difference and it represents an infinitely small width of x when it becomes very, very, very small. $\int f\, dx$ indicates the sum of an infinite number of rectangles, each with a width that is infinitely small.

Constant: A special, unchanging number that arises naturally in mathematics. An example is the number 32. It is known as the **constant of acceleration**. The opposite of a constant is a variable. (See: Variable)

Constant of integration: A constant that expresses the uncertainty in the process of finding antiderivatives.

Critical point: Also known as the "stationary point," this is a point where the derivative of a function is equal to 0. The tangent line at that point is horizontal. (See: Optimization)

Definite integral: An integral that is evaluated over an interval, written as $\int_a^b f(x)dx$.

Definite integration does not involve a constant of integration and gives a definite value—unlike *in*definite integration, which gives a function.

Dependent variable: A variable whose value depends on the value of another variable. In the function y=f(x), y is the dependent variable. (See: Independent variable)

Derivative: The derivative will tell you how fast you're going at any instant in time. It also represents the slope of the tangent line that touches a curve at one particular point. The derivative is written as $\frac{dy}{dt}$, where dy represents what happens when Δy approaches 0 and dt represents what happens to Δt as it approaches 0.

Differential equation: An equation that contains a derivative and describes how a system changes after a short period of time. $\frac{dy}{dx} = 4$ is a differential equation. We can write many laws of physics, biology, chemistry, and economics as differential equations.

Equation: A mathematical statement used to show that two amounts are equal. For example, 1+1=2. A quadratic equation is one that contains a variable raised to the second power (e.g., $y=16t^2$). Here, y and t are variables and 16 is a constant.

Exponent: The number that tells how many times the base number is multiplied by itself. For $y=16t^2$, the exponent is 2. The exponent is also known as the logarithm.

Free fall motion: Free fall motion is the motion of an object under the effect of gravitational forces only. Over 300 years ago, Galileo discovered that objects under free fall motion have a changing velocity and a constant acceleration of 32 ft/sec². We find the distance traveled in free fall with the equation $y=16t^2$. The velocity is v=32t.

Function: A function is a relationship between variables. It is a mathematical expression that changes one number into another. If x and y are variables, and if every value of x is associated with exactly one value of y, then y is said to be a function of x. y = f(x). y is called the dependent variable (its value depends on what is chosen for x) and x is called the independent variable. The term “a function of” could also be called a “depend-upon” relation.

Fundamental theorem of calculus: The theorem connects integrals as sums, with integrals as antiderivatives. So, finding the slope of a curve (differentiation) is the inverse operation of finding the area under a curve (antidifferentiation).

Expressed simply as: $\frac{dA}{dx} = y$ A $=\int y\,dx$ A = Area (see Appendix 5)

Graph: A diagram showing the relation between variable quantities—typically of two variables, each measured along one of a pair of axes at right angles.

Independent variable: The variable that determines the value of the dependent variable. For example, for y = f(x), x is the independent variable. (See: Dependent variable)

Instantaneous velocity: Velocity is the rate of change with respect to time. Instantaneous velocity is the rate at which an object is moving at a particular moment. We find it by taking the derivative of position and write it as $\frac{dy}{dt}$.

Integral: The process of doing the inverse of differentiation is called integration. The result is called the integral and represents the area under a curve. The integral sign is $\int$, a tall S, standing for "sum."

Integration: Integration is a procedure for finding a distance equation when its derivative is given. In differentiation, we break up things into smaller and smaller parts. In integration, we add up all the smaller parts. Integration is a summation and shown symbolically as $\int$dy = y. This means that the sum of many little bits of y add up to y.

Integration (or antidifferentiation) is the procedure for finding the distance equation when its derivative is known.

You can also use **integration** to find the area under a curve between set boundaries. You do this using repeated width reduction of inscribed rectangles, then by adding the area of these rectangles to get an accurate result. Remember that the width of each rectangle *never* equals zero!

Integration results in two antiderivatives—one subtracted from the other. This process finds the area.

Instantaneous rate of change: The rate of change at a particular moment or point on a curve. This is the same as the value of the derivative at a particular point. For a function, the instantaneous rate of change at a point is the same as the slope of the tangent line. We write this as $\frac{dy}{dx}$. (See: Instantaneous velocity)

Instantaneous velocity: The rate at which an object is moving at a particular moment. It is the same as the derivative of the function describing the position of the object at a particular time. We write it as $\frac{dy}{dt}$. (See: Instantaneous rate of change)

Integrand: A function that is to be integrated. In $\int f(x)$dx, f(x) is the integrand.

Inverse operations: Two operations are said to be the "inverse" (opposite) of each other if the effect of one is canceled by the other. Addition is the inverse of subtraction, and division is the inverse of multiplication. Finding the slope of a curve (differentiation) is the inverse operation of finding the area under a curve (integration).

Limit: A limit is the value that a function or sequence approaches as the input approaches some value. We use limits to define derivatives. For f(x) $= \frac{1}{x}$, the limit as x approaches 3 is ⅓ .

Limit of integration: For the definite integral $\int_a^b f(x)dx$, the limits of integration are numbers represented by a and b. This represents the boundaries of the area being formed.

Linear function: A linear function is one with a constant growth rate or slope. The formula for a linear function is y=mx+b. (See Appendix 3)

Maximum: In Calculus, a maximum is a high point on a curve where the graph shifts from increasing to decreasing. Here, the tangent line has a slope of zero and the derivative of the equation representing the curve, likewise, has a value of zero.
To find the maximum height of a curve, perform the following steps:

1. Find the derivative.
2. Find the point when the derivative is equal to zero. Do this by setting the resulting equation equal to zero and solving for the variable.
3. Plug this variable back into the original equation to find the maximum height.

Optimization: A method used to find the absolute maximum or minimum value of the function.

Power rule: The formula for finding the derivative and integral of a power of a variable, expressed as $\frac{dx^n}{dx} = nx^{n-1}$.
Examples:
$\frac{dx^3}{dx} = 3x^2$ $\int x^n dx = \frac{1}{n+1}x^{n+1}$ $\int x^3 dx = \frac{x^4}{4}$

Proportional: A relationship that is related or corresponds to another function. An example is an equation that shows two ratios are equal—the number of apples in a crop is proportional to the number of trees in the orchard.

Quadratic equation: An equation where the highest exponent of the variable is 2. It will look something like x^2—but not x^3, etc. When graphed, a quadratic equation produces a curve known as a parabola.

Rate: A comparison or ratio of two numbers or variables. For example, rate is the ratio between distance and time, so $\frac{dy}{dt}$= the rate at which y changes. Acceleration is the rate an object's velocity changes with time. For example, $4t^2$ grows at the rate of 8t.

Rate of change (ROC): A change that takes place in a unit of time (though ROC without reference to time is sometimes considered). A constant ROC means no acceleration.

Second derivative: The second derivative of a function (f) is the derivative of the derivative of (f). Acceleration is the second derivative of distance. For example, s = distance, v = velocity, a=acceleration. Then, v =$\frac{ds}{dt}$, a =$\frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Slope of a line or curve: A number used to indicate the steepness of a line.
The slope of a curve at a point is defined as the slope of the tangent line that touches the curve at that point. The slope of a curve is always changing. To find the slope at any point on a curve means to find the derivative at that point. The slope is the derivative.

Speed: An absolute value of velocity. Speed plus direction is called velocity.

Tangent line: A line that touches a curve at just one point.

Variable: A symbol that may assume any number of values and is usually represented by a letter such as x, y , s, or t. A variable is the opposite of a constant. In the statement x-18, x is the variable. (See: Constant)

Velocity: The rate of change of position with respect to time. It has both size and direction. We find velocity by taking the derivative of the position function. For example, if y represents the number of bacteria in a dish, then Δy is the change in the number of bacteria over a period of time, t, written as Δt. So, we write the average growth rate as $\frac{\Delta y}{\Delta t}$. Velocity is the derivative of position. Velocity changing in time is called acceleration. *(See: Instantaneous Rate of change, Instantaneous velocity, Speed, and Acceleration)*

Quiz answers

1. T
2. B
3. C
4. C
5. T
6. A
7. B
8. T
9. C
10. A
11. C
12. 5t evaluated from (t=3 - t=1) = 10
13. 96 ft/sec, derivative
14. 5+32t
15. $8x^7$
16. 3 seconds
17. 7/3
18. $\frac{x^2}{2} + C$, B
19. 128 ft/sec
20. $\frac{x^2}{2} + C$
22. v = 32t, $\frac{dv}{dt}$ = acceleration = 32
23. There needs to be a constant of integration indicated by a +C.

Index

References, Ideas, and Further Reading

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“Too much concentration on the mechanics of problem solving can degrade the teaching of calculus into an empty drill.”
—Richard Courant, late Professor of Mathematics at New York University

“Mathematics is taught as an isolated subject with few, if any, ties to the real world. To students, mathematics appears to deal almost entirely with things which are of no concern at all to man.”
—Morris Kline, late Professor of Mathematics

“The nature of the derivative is perhaps best understood by thinking in terms of velocity, much as Newton did.”
—Morris Kline, late Professor of Mathematics

Calculus was developed in the seventeenth century to study four types of math problems of that time:

- Find the **slope** of a line that just touches a curve at a point.
- Find the **area** of a region below a curve.
- Find the maximum or minimum value of a quantity.
- Knowing the formula for the distance traveled by a moving body, find the velocity and acceleration at any instant. Then in reverse, given the velocity or acceleration at any instant, find the distance traveled by the body in a specified period of time.

Analyzing the free falling motion of a roller will help in understanding the above ideas. Repeated use of two equations developed by Galileo are used to define the Calculus concepts of **Differentiation** and **Integration.** In simple terms, Calculus is just about **Slopes** and **Areas.**

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